ON HEAT CONDUCTION AND WAVE PROPAGATION IN RIGID SOLIDS

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Abstract. This paper is concerned with conduction of heat and the related problem of propagation of thermal waves in stationary rigid solids. Special attention is given to rate-dependent response and the ensuing conditions of propagation in the conducting medium. By considering small time-dependent temperature variations superposed on a finite nonuniform equilibrium temperature field, certain conclusions are reached which also shed light on the so-called "second sound" phenomenon.

1. Introduction

This paper is concerned with heat conduction and propagation of thermal waves in rigid solids. Our approach and point of view is that of recent developments in continuum thermodynamics. We employ the balance of energy and the (Clausius-Duhem) entropy production inequality, as well as constitutive assumptions characterizing the local thermal behavior of the material. For example, if temperature, temperature gradient and rate of temperature are taken as independent variables, then the thermal variables such as specific internal energy, specific entropy and heat flux are determined by constitutive equations. When the constitutive assumptions include rate of temperature as an independent variable there arises a local production of entropy (see Eq. (2.2)) in addition to the entropy production due to conduction, which is inherent in the usual rate-independent response. The rate-dependent thermal response studied here and the corresponding generalization of the classical (Fourier) heat

conduction equation leads to a prediction of finite speed thermal waves, as well as to other novel phenomena associated with local thermal dissipation.

Before describing the scope of the paper in detail, it is desirable to recall for background information certain other developments which bear on conduction of heat in rigid solids. Within the framework of modern developments in continuum physics, attention was largely confined until recently to purely mechanical constitutive equations. Arguments for obtaining necessary and sufficient conditions for the validity of the entropy production inequality, limited to nonlinear elastic materials with heat conduction and viscosity, were given by Coleman and Noll [1]. A more extensive discussion of generalizations of the classical theory of linearly viscous fluids with linear heat conduction, in which the independent thermodynamic variables are temperature and temperature gradient, is contained in a paper by Coleman and Mizel [2]. Among subsequent developments, we mention Coleman's work [3] on thermodyanmics of materials with fading memory, in which the thermo-mechanical constitutive equations are assumed to be functionals over the time histories of the chosen independent variables. Similar constitutive assumptions are utilized by Coleman and Gurtin [4] in their investigation of the thermal behavior of rigid heat conductors.

Certain types of well-known mechanical behavior — typical of materials or rate-type — such as that for linear (Newtonian) viscous fluids are appropriately characterized by rate-dependent functions rather than by

memory functionals.* Likewise, we may expect certain thermal behavior of rigid conductors to be best described by rate-dependent functions and this is the underlying premise of the generalization of the classical heat conduction equation sought here.

That a generalization of the classical heat conduction equation for solids is desirable has been recognized for several years. This view is based primarily on the fact that the classical equation is parabolic and does not admit the possibility of finite speeds of propagation of thermal pulses. Most of the previous attempts to alter the heat conduction equation so as to predict thermal waves, often referred to in the literature as "second sound," have been based on an <u>ad hoc</u> generalization of Fourier's heat conduction law; see for example Chester [5] and Ulbrich [6]. This generalization, referred to by Ulbrich as Vernotte's hypothesis [7], appears as

$$\underset{\sim}{q} + \tau \dot{\underset{\sim}{q}} = k \operatorname{grad} \theta \tag{1.1}$$

*We may recall here that constitutive equations of linear viscous fluids can be regarded as exact constitutive relations which describe the behavior of a class of fluids in all motions. On the other hand, in a functional theory (e.g., of the type developed by Coleman [3]), a linearly viscous material approximates a general material with fading memory only in the limit of "slow" motions.

for rigid conductors, where $\underset{\sim}{q}$, $\dot{\underset{\sim}{q}}$, θ represent the heat flux, the time rate of heat flux, and temperature, while τ, k are material constants. With reference to (1.1) Ulbrich [6] states "Although Vernotte's proposed revision of Fourier's hypothesis adequately circumvents the paradox of infinite velocity, no apparent physical justification can be offered for the addition of the second term ..."

A different approach from that based on (1.1), but with the same objective, is taken by Kaliski [8]. He starts with the assumption that the heat conduction equation should be a second order hyperbolic equation and then generalizes Onsager's reciprocal relations and the associated form of the entropy inequality so as to accommodate this assumption.

We also draw attention to a very recent paper by Gurtin and Pipkin [9] whose work is motivated by (1.1). They develop a nonlinear theory for rigid heat conductors in which the constitutive assumptions are in the form of functionals over the temperature history. They show that an equation of the form of (1.1) is a special case of their linearized constitutive equation for the heat flux $\underset{\sim}{q}$. Their associated generalized heat conduction equation in the linear theory is given by

$$c\ddot{\theta}(\underset{\sim}{x},t) + \beta(0)\dot{\theta}(\underset{\sim}{x},t) + \int_0^\infty \beta'(s)\dot{\theta}(\underset{\sim}{x},t-s)ds$$

$$= a(0)\Delta\theta(\underset{\sim}{x},t) + \int_0^\infty a'(s)\Delta\theta(\underset{\sim}{x},t-s)ds + \dot{r}(\underset{\sim}{x},t) \quad , \tag{1.2}$$

where $a(s)$, $\beta(s)$ are called the heat flux relaxation function and the energy relaxation function. This equation is in a form that predicts finite speeds of propagation of thermal waves. An unusual feature of (1.2), however, is the presence of $\dot{r}$ (rather than r) which evidently is due to the fact that (1.2) is obtained from the first time derivative of the energy equation rather than from the energy equation itself.

In the present paper, which is concerned with rigid stationary conductors, in general we use direct (coordinate free) notation. Thus, vectors and points in three-dimensional Euclidean space are denoted by boldface Latin lower case letters while boldface Latin capital letters are used to designate tensors of order two; also, Greek lower case letters are reserved for scalar thermodynamic variables. An exception to the above notation occurs when we write the energy equation as a partial differential equation in the temperature. Then, for ease of comparison and analysis, all quantities in the energy equation are expressed in terms of their Cartesian components.

In section 2, following some preliminaries, for clarity and later comparison we include a discussion of nonlinear constitutive equations for rate-independent response** when temperature and temperature gradient are taken as independent variables. A generalization, in which the independent variables include the rate of temperature, is introduced and developed in section 3. This leads to a local thermal dissipation of energy and to the possibility of temperature changes without heat flow. The appropriate "heat conduction equation" is then derived and is used

**These constitutive equations when properly linearized reduce to those appropriate to the classical theory of heat conduction.

in section 4 to investigate the existence of finite speeds of propagation of second order discontinuities in the temperature.

In order to arrive at more explicit conclusions, we specialize in section 5 the results of section 3 by assuming that the dependence on the temperature rate is linear. We then re-examine the energy equation for the case of isotropic materials with a center of symmetry and, in particular, make certain observations regarding the speed of propagation of thermal waves in one space dimension. Of special significance is the result in section 5 that real wave speeds are possible only if there is a sufficiently large temperature gradient present in the medium through which the wave is to propagate. Qualitative support for this phenomenon based on predictions from microscopic theories of heat transport are cited from the literature.

Finally, motivated partially by the results of section 5, we derive in section 6 a linear heat conduction equation governing infinitesimal time-dependent temperature variations superposed on a finite nonuniform equilibrium temperature change from the reference temperature. The non-uniformity of the equilibrium temperature change is essential here; the resulting linear equation can give rise to finite wave speeds and appears to explain the "second sound" phenomenon. The predicted thermal waves are such that they may travel in only one direction relative to the temperature gradient – either with or against – depending on the sign of a certain material constant. This is similar to the relation between the relative directions of the heat flux and temperature gradient vectors.

2. Preliminaries. Rate-independent response.

Let $\underset{\sim}{x}$ be the (three dimensional) position vector relative to a fixed rectangular Cartesian coordinate system and let t denote time. Since we shall be concerned only with rigid stationary conductors, no distinction between material particles and their positions in space is necessary. Let ϵ, η, θ and $\underset{\sim}{q}$ denote the internal energy per unit mass, the entropy per unit mass, the temperature (assumed positive), and the heat flux vector, all functions of $\underset{\sim}{x}, t$ in a prescribed domain. These quantities, together with the heat supply per unit mass r, must satisfy for each $\underset{\sim}{x}, t$ the energy equation+

$$- \operatorname{div} \underset{\sim}{q} + \rho r = \rho \dot{\epsilon} \quad , \tag{2.1}$$

and the entropy production inequality which, for later convenience, we write as

$$\begin{gathered} \gamma = \gamma_{loc.} + \gamma_{con.} \geqq 0 \quad , \\ \rho \gamma_{loc.} = \rho \dot{\eta} + \frac{1}{\theta} \operatorname{div} \underset{\sim}{q} - \rho \frac{r}{\theta} \quad , \quad \rho \gamma_{con.} = - \frac{1}{\theta^2} \underset{\sim}{q} \cdot \underset{\sim}{g} \quad , \end{gathered} \tag{2.2}$$

where $\gamma_{loc.}$ and $\gamma_{con.}$ represent the local entropy production and the production of entropy due to conduction, the abbreviation $\underset{\sim}{g} = \operatorname{grad} \theta$ stands for the temperature gradient, a superposed dot denotes partial

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+These local forms can be deduced from the corresponding statements in integral form. See, for example Truesdell and Noll [10]; our $\underset{\sim}{q}$, r correspond to their $-\underset{\sim}{h}$, q .

differentiation with respect to t, and ρ is the mass density, a function of $\underset{\sim}{x}$ only for rigid stationary bodies. Combining (2.1) and (2.2), we obtain the inequality

$$-\rho\dot{\epsilon} + \rho\theta\dot{\eta} - \frac{1}{\theta}\underset{\sim}{q}\cdot\underset{\sim}{g} \geqq 0 \quad . \tag{2.3}$$

In terms of the free energy function ψ defined by

$$\psi = \epsilon - \eta\theta \quad , \tag{2.4}$$

the energy equation (2.1) and the inequality (2.3) can be expressed in the alternative forms

$$-\operatorname{div}\underset{\sim}{q} + \rho r = \rho(\dot{\psi} + \eta\dot{\theta} + \dot{\eta}\theta) \quad , \tag{2.5}$$

and

$$-\rho\dot{\psi} - \rho\eta\dot{\theta} - \frac{1}{\theta}\underset{\sim}{q}\cdot\underset{\sim}{g} \geqq 0 \quad . \tag{2.6}$$

Taking as independent variables θ and $\underset{\sim}{g}$ we consider in this section a material whose response is characterized by rate-independent constitutive assumptions of the form++

$$\psi = \psi(\theta,\underset{\sim}{g}) \quad , \quad \eta = \eta(\theta,\underset{\sim}{g}) \quad , \quad \underset{\sim}{q} = \underset{\sim}{q}(\theta,\underset{\sim}{g}) \quad . \tag{2.7}$$

++The same symbols can be used here for a function and its value without confusion.

The introduction of (2.7) into (2.6) results in

$$-\rho(\frac{\partial\psi}{\partial\theta} + \eta)\dot{\theta} - \rho(\frac{\partial\psi}{\partial \underset{\sim}{g}}) \cdot \dot{\underset{\sim}{g}} - \frac{1}{\theta}\underset{\sim}{q} \cdot \underset{\sim}{g} \geq 0 \quad . \tag{2.8}$$

With θ, $\underset{\sim}{g}$, $\dot{\theta}$ fixed, all terms in (2.8) are determined except $\dot{\underset{\sim}{g}}$, which may assume arbitrary values. Then, in order to satisfy (2.8), we must have

$$\frac{\partial\psi}{\partial \underset{\sim}{g}} = \underset{\sim}{0} \quad , \quad \psi = \psi(\theta) \quad , \tag{2.9}$$

and (2.8) reduces to

$$-\rho(\frac{\partial\psi}{\partial\theta} + \eta)\dot{\theta} - \frac{1}{\theta}\underset{\sim}{q} \cdot \underset{\sim}{g} \geq 0 \quad . \tag{2.10}$$

Using (2.10) and repeating the above argument with θ, $\underset{\sim}{g}$ fixed and $\dot{\theta}$ arbitrary we find

$$\eta = -\frac{\partial\psi}{\partial\theta} = \eta(\theta) \quad , \quad -\frac{1}{\theta}\underset{\sim}{q} \cdot \underset{\sim}{g} \geq 0 \quad . \tag{2.11}$$

Thus the entropy is determined by the free energy, a function of temperature only, and according to $(2.11)_2$ heat cannot flow in a direction of increasing temperature.

Let the symmetry group of the material be defined by the set of all time-dependent orthogonal second order tensors $\underset{\sim}{A}$. Then, under a change of reference frame characterized by $\underset{\sim}{A}$, the scalars ϵ, η, θ, ψ are unaffected but the vectors $\underset{\sim}{q}$, $\underset{\sim}{g}$ transform according to

$$\underset{\sim}{q} \to \underset{\sim}{A}\,\underset{\sim}{q} \quad , \quad \underset{\sim}{g} \to \underset{\sim}{A}\,\underset{\sim}{g} \quad . \tag{2.12}$$

The constitutive equations (2.7) are invariant under such a change of frame if and only if for each $\underset{\sim}{A}$ and all values of $\underset{\sim}{q}$, $\underset{\sim}{g}$,

$$\underset{\sim}{A}\,\underset{\sim}{q}(\theta,\underset{\sim}{g}) = \underset{\sim}{q}(\theta,\underset{\sim}{A}\,\underset{\sim}{g}) \quad . \tag{2.13}$$

If, in particular, the material has a center of symmetry the identity (2.13) must be satisfied also for $\underset{\sim}{A} = -\underset{\sim}{1}$, where $\underset{\sim}{1}$ is the unit tensor, so that

$$-\underset{\sim}{q}(\theta,\underset{\sim}{g}) = \underset{\sim}{q}(\theta,-\underset{\sim}{g}) \quad . \tag{2.14}$$

It follows that if $\underset{\sim}{q}$ is a continuous function of $\underset{\sim}{g}$ at $\underset{\sim}{g} = \underset{\sim}{0}$, then

$$\underset{\sim}{q}(\theta,\underset{\sim}{0}) = \underset{\sim}{0} \quad , \tag{2.15}$$

i.e., there can be no heat flow in the absence of a temperature gradient.*

From the energy equation (2.5) with (2.7), (2.9) and (2.11) there follows in component form

$$\left(\frac{\partial q_k}{\partial g_\ell}\right)\theta_{,\ell k} + \left(\frac{\partial q_k}{\partial \theta}\right)\theta_{,k} - \rho\theta\left(\frac{\partial^2\psi}{\partial \theta^2}\right)\dot{\theta} = \rho r \quad , \tag{2.16}$$

where q_k, $g_\ell (\equiv \theta_{,\ell})$, are the Cartesian components of $\underset{\sim}{q}$, $\underset{\sim}{g}$, a comma

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*Similar arguments have appeared many times in the literature. See, for example [2].

preceding an index denotes partial differentiation with respect to rectangular Cartesian coordinates x_k $(k=1,2,3)$, and the usual summation is implied by repeated indices. When $\underset{\sim}{q}$, ψ in $(2.7)_3$, $(2.9)_2$ are given explicit representations in terms of θ, $\underset{\sim}{g}$, (2.16) provides a quasi-linear partial differential equation to be satisfied by θ. From (2.16), together with (2.15), it follows that when the heat supply r vanishes in rigid conductors having the rate-independent type thermal response (2.7) there can be no time dependence in the temperature field without a corresponding spatial dependence,[+] i.e., there can be no change in temperature except that resulting from the flow of heat. Correspondingly, in view of $(2.11)_2$ and $(2.2)_3$, the entropy production is due to conduction alone and there is no local entropy production.

The residual energy equation (2.16) is second order in its spatial derivatives and only first order in its time derivative and therefore clearly cannot predict thermal waves (propagating second order discontinuities) with real finite wave speeds. This will be elaborated upon in sections 4-6.

We close this section by noting that the classical linear Fourier heat conduction equation results from the linearization of (2.16) when $(2.7)_3$ takes the special form represented by the Fourier law

$$\underset{\sim}{q}(\theta,\underset{\sim}{g}) = -\underset{\sim}{K}(\theta)\underset{\sim}{g} \tag{2.17}$$

and $\psi(\theta)$ in $(2.9)_2$ is assumed to be polynomial.

[+]Assuming, of course, that $\dfrac{\partial^2\psi}{\partial\theta^2}$ does not vanish.

3. Dependence on temperature rate.

We consider here a generalization of the constitutive assumptions (2.7) by including the first time derivative of temperature, as well as the temperature and its gradient, as independent variables. Thus, instead of (2.7), we now assume

$$\psi = \psi(\theta, \underset{\sim}{g}, \dot{\theta}) \quad , \quad \eta = \eta(\theta, \underset{\sim}{g}, \dot{\theta}) \quad , \quad \underset{\sim}{q} = \underset{\sim}{q}(\theta, \underset{\sim}{g}, \dot{\theta}) \quad , \tag{3.1}$$

so that the inequality (2.6) becomes

$$-\rho\left(\frac{\partial\psi}{\partial\theta} + \eta\right)\dot{\theta} - \rho\left(\frac{\partial\psi}{\partial \underset{\sim}{g}}\right) \cdot \dot{\underset{\sim}{g}} - \rho\left(\frac{\partial\psi}{\partial\dot{\theta}}\right)\ddot{\theta} - \frac{1}{\theta}\, \underset{\sim}{q} \cdot \underset{\sim}{g} \geqq 0 \quad . \tag{3.2}$$

With θ, $\underset{\sim}{g}$, $\dot{\theta}$ fixed it follows from (3.1) that all terms in (3.2) are determined except $\ddot{\theta}$, which may assume arbitrary values. In order for (3.2) to be satisifed for all $\ddot{\theta}$, its coefficient $\left(\frac{\partial\psi}{\partial\dot{\theta}}\right)$ must vanish, i.e., ψ must be independent of $\dot{\theta}$. Similarly ψ cannot depend on $\underset{\sim}{g}$, so that $(3.1)_1$ and (3.2) reduce to

$$\psi = \psi(\theta) \tag{3.3}$$

and

$$-\rho\left(\frac{\partial\psi}{\partial\theta} + \eta\right)\dot{\theta} - \frac{1}{\theta}\, \underset{\sim}{q} \cdot \underset{\sim}{g} \geqq 0 \quad , \tag{3.4}$$

respectively.

Next we decompose η into two parts $\eta^{(o)}$, $\eta^{(e)}$ defined by*

$$\eta(\theta,\underset{\sim}{g},\dot{\theta}) = \eta^{(o)}(\theta)+\eta^{(e)}(\theta,\underset{\sim}{g},\dot{\theta}) \quad , \quad \eta^{(o)}(\theta)=\eta(\theta,\underset{\sim}{0},0) \quad . \tag{3.5}$$

Assuming that η is continuous in $\dot{\theta}$ at $\dot{\theta}=0$, we have** in the limit as $\alpha \to 0$

$$\eta(\theta,\underset{\sim}{g},\alpha\dot{\theta}) = \eta(\theta,\underset{\sim}{g},0) + o(1) \quad , \tag{3.6}$$

so that

$$\eta^{(e)}(\theta,\underset{\sim}{0},\alpha\dot{\theta}) = o(1) \quad , \tag{3.7}$$

where α is a real number. Now in (3.4) put $\underset{\sim}{g} = \underset{\sim}{0}$, replace $\dot{\theta}$ by $\alpha\dot{\theta}$ and use (3.5), (3.7) to obtain

$$-\rho(\frac{\partial\psi}{\partial\theta}+\eta^{(o)})\alpha\dot{\theta} + o(\alpha) \geqq 0 \quad , \tag{3.8}$$

where $\lim\limits_{\alpha\to 0} \frac{o(\alpha)}{\alpha} = 0$. It then follows from (3.8) that

$$\eta^{(o)} = -\frac{\partial\psi}{\partial\theta} \quad , \tag{3.9}$$

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*It should be noted that the decomposition of η in (3.5) does not imply that $\eta^{(o)}$ is independent of t. Here θ, $\underset{\sim}{g}$, $\dot{\theta}$ are considered as independent variables. Alternatively $\tilde{\eta}^{(o)}(\theta)$ is the limit of $\eta(\theta,\alpha\underset{\sim}{g},\beta\dot{\theta})$ as $\alpha,\beta \to 0$.

**A function of α defined in a neighborhood of α_o is $o(1)$ or $O(1)$ as $\alpha \to \alpha_o$ according as it is zero or bounded in this limit.

and the inequality (3.4) assumes the form

$$-\rho\eta^{(e)}\dot{\theta} - \frac{1}{\theta}\,\underset{\sim}{q}\cdot\underset{\sim}{g} \geq 0 \quad . \tag{3.10}$$

In view of (3.1), (3.3), (3.5) and (3.9) the energy equation (2.5) becomes (in component form)

$$\left(\frac{\partial q_k}{g_\ell}\right)\theta_{,\ell k} + \left(\frac{\partial q_k}{\partial\dot{\theta}} + \rho\theta\,\frac{\partial\eta^{(e)}}{\partial g_k}\right)\dot{\theta}_{,k} + \rho\theta\left(\frac{\partial\eta^{(e)}}{\partial\dot{\theta}}\right)\ddot{\theta}$$

$$+ \left(\frac{\partial q_k}{\partial\theta}\right)\theta_{,k} + \rho\left(\eta^{(e)} + \theta\,\frac{\partial\eta^{(e)}}{\partial\theta} - \theta\,\frac{\partial^2\psi}{\partial\theta^2}\right)\dot{\theta} = \rho r \quad , \tag{3.11}$$

which is the counterpart of (2.16) under the assumptions (3.1)$^{+}$.

Many of the conclusions drawn in section 2 are altered for the rate-dependent response under consideration. In particular, even though ψ still reduces to a function of θ only, we cannot conclude the same about η which remains dependent on $\underset{\sim}{g}$ and $\dot{\theta}$. Furthermore, it is clear from (2.2) and (3.10) that the inclusion of $\dot{\theta}$ as an independent variable in constitutive assumptions (3.1) has led to a local entropy production. In addition, the total entropy is no longer expressible in terms of the free energy as in the case not only of the rate-independent

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$^{+}$It is worth noting that if in place of (3.1) we had assumed $\psi = \psi(\theta,\underset{\sim}{g},\dot{\theta},\dot{\underset{\sim}{g}})$, etc., that is, if the rate of the temperature gradient had been included as an independent variable also, then (3.3) would be replaced by $\psi = \psi(\theta,\underset{\sim}{g})$. This should be contrasted with the results obtained by Coleman and Mizel [11], where is it shown that ψ cannot depend on temperature gradients when a dependence on rates is excluded. See also in this regard Eringen [12] where the rate of temperature and rates of temperature gradients are included as independent variables in the context of a general thermo-mechanical investigation.

type response but also of the more general response characterized by a functional over the time history of temperature as investigated by Coleman and Gurtin [4].

The requirements that the constitutive equations (3.3) and $(3.1)_{2,3}$ satisfy the appropriate transformation relations under a change of reference frame (imposed by the symmetry group of the material), that the heat flux function $\underset{\sim}{q}$ be continuous in $\underset{\sim}{g}$ at $\underset{\sim}{g} = \underset{\sim}{0}$ and that the material possess a center of symmetry imply (with an argument parallel to that used to arrive at (2.15)) that $\underset{\sim}{q}$ must be an odd, and η an even, function of $\underset{\sim}{g}$ with

$$\underset{\sim}{q}(\theta,\underset{\sim}{0},\dot{\theta}) = \underset{\sim}{0} \quad . \tag{3.12}$$

Therefore, as in the rate-independent case, there can be no heat flux corresponding to zero temperature gradient. We also observe the inequalities

$$\underset{\sim}{q}(\theta,\underset{\sim}{g},0) \cdot \underset{\sim}{g} \leqq 0 \quad , \quad \eta^{(e)}(\theta,\underset{\sim}{0},\dot{\theta})\dot{\theta} \leqq 0 \quad , \tag{3.13}$$

which follow from (3.10), but we note that $\underset{\sim}{q} \cdot \underset{\sim}{g} \leqq 0$ cannot in general be concluded.

4. Propagation of thermal waves

We investigate here the possibilities of the existence of propagating second order discontinuities, i.e., discontinuities in the second derivatives, in the temperature field. For convenience, however, we confine our analysis to one space dimension x. Writing q, g for the x-components of $\underset{\sim}{q}$, $\underset{\sim}{g}$, we obtain from (3.11) its one-dimensional counterpart

$$\left(\frac{\partial q}{\partial g}\right)\theta_{,xx} + \left(\frac{\partial q}{\partial \dot{\theta}} + \rho\theta\,\frac{\partial \eta^{(e)}}{\partial g}\right)\dot{\theta}_{,x} + \rho\left(\theta\,\frac{\partial \eta^{(e)}}{\partial \dot{\theta}}\right)\ddot{\theta}$$
$$+ \left(\frac{\partial q}{\partial \theta}\right)\theta_{,x} + \rho\left(\eta^{(e)} + \theta\,\frac{\partial \eta^{(e)}}{\partial \theta} - \theta\,\frac{\partial^2 \psi}{\partial \theta^2}\right)\dot{\theta} = \rho r \quad . \tag{4.1}$$

We assume that θ, g, $\dot{\theta}$ and r are continuous functions of x, t and denote by [ ] the value at the propagating discontinuity of the jump of whatever quantity it brackets. If we allow discontinuities in second order derivatives of θ and assume that the first derivatives are continuous, there follows from (4.1)

$$\left(\frac{\partial q}{\partial g}\right)[\theta_{,xx}] + \left(\frac{\partial q}{\partial \dot{\theta}} + \rho\theta\,\frac{\partial \eta^{(e)}}{\partial g}\right)[\dot{\theta}_{,x}] + \rho\left(\theta\,\frac{\partial \eta^{(e)}}{\partial \dot{\theta}}\right)[\ddot{\theta}] = 0 \quad . \tag{4.2}$$

Using standard jump conditions* we have

*See Thomas [13] or Truesdell and Toupin [14; Sec. 181].

$$[\dot{\theta}_{,x}] = -U[\theta_{,xx}] \quad , \quad [\ddot{\theta}] = U^2[\theta_{,xx}] \quad , \tag{4.3}$$

where U is the speed of propagation of the discontinuity. Substitution of (4.3) into (4.2) gives

$$U = \frac{\frac{\partial q}{\partial \dot{\theta}} + \rho\theta \frac{\partial \eta^{(e)}}{\partial g} \pm \left[\left(\frac{\partial q}{\partial \dot{\theta}} + \rho\theta \frac{\partial \eta^{(e)}}{\partial g}\right)^2 - 4\rho\theta \frac{\partial q}{\partial g} \frac{\partial \eta^{(e)}}{\partial \dot{\theta}}\right]^{1/2}}{2\rho\theta \frac{\partial \eta^{(e)}}{\partial \dot{\theta}}} \quad , \tag{4.4}$$

provided $[\theta_{,xx}]$ and $\frac{\partial \eta^{(e)}}{\partial \dot{\theta}}$ are not zero. In the event $\frac{\partial \eta^{(e)}}{\partial \dot{\theta}}$ does vanish, (4.2) and (4.3) yield

$$U = \frac{\frac{\partial q}{\partial g}}{\frac{\partial q}{\partial \dot{\theta}} + \rho\theta \frac{\partial \eta^{(e)}}{\partial g}} \quad . \tag{4.5}$$

The terms on the right hand side of (4.4), (4.5) are evaluated at the propagating discontinuity so that U can be considered alternatively as a function of x or t only.**

The following observations are apparent from (4.4) and (4.5):

(i) Equation (4.4) shows that second order discontinuities in temperature, i.e., thermal waves, may possibly occur with two distinct real wave speeds. This depends, of course, on the relative magnitudes and signs of the terms in (4.4) and will be investigated further for particular

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** An alternate derivation of (4.4) results from the characteristic relation appropriate to the quasi-linear partial differential equation (4.1). It is thereby seen that discontinuities of the type under consideration can occur only along a characteristic. See Courant and Hilbert [15].

representations of $\eta^{(e)}$, ψ and q in the following sections.

(ii) It follows from (4.5) that if $\eta^{(e)}$ is independent of $\dot{\theta}$, then only one wave speed is possible. In fact we see that thermal waves may occur when $\eta^{(e)}$ vanishes altogether. Thus, thermal dissipation due to a local entropy production, represented by the first term of the inequality in (3.10), may not be necessary for thermal waves, provided q remains dependent on $\dot{\theta}$ as well as g. Equation (4.5) also shows that when $\eta^{(e)}$ vanishes and q does not depend on $\dot{\theta}$, as in the rate-independent response discussed in section 2, no finite speed thermal wave can occur.

5. <u>Linearity in temperature rate</u>

Recalling $(3.1)_{2,3}$ and (3.3) we introduce in this section specific assumptions regarding the dependence of the response functions on the temperature rate. Before doing so we decompose $\underset{\sim}{q}$ (as was done with η in (3.5)) into two parts $\underset{\sim}{q}^{(o)}$, $\underset{\sim}{q}^{(e)}$ defined by

$$\underset{\sim}{q}(\theta,\underset{\sim}{g},\dot{\theta}) = \underset{\sim}{q}^{(o)}(\theta,\underset{\sim}{g}) + \underset{\sim}{q}^{(e)}(\theta,\underset{\sim}{g},\dot{\theta}) \quad , \quad \underset{\sim}{q}^{(o)}(\theta,\underset{\sim}{g}) = \underset{\sim}{q}(\theta,\underset{\sim}{g},0) \quad . \tag{5.1}$$

Then (3.12) implies

$$\underset{\sim}{q}(\theta,\underset{\sim}{0},\dot{\theta}) = \underset{\sim}{q}^{(o)}(\theta,\underset{\sim}{0}) + \underset{\sim}{q}^{(e)}(\theta,\underset{\sim}{0},\dot{\theta}) = \underset{\sim}{0} \quad . \tag{5.2}$$

Since $\underset{\sim}{q}^{(e)}(\theta,\underset{\sim}{g},0)$ vanishes by (5.1), it follows that

$$\underset{\sim}{q}(\theta,\underset{\sim}{0},0) = \underset{\sim}{q}^{(o)}(\theta,\underset{\sim}{0}) = \underset{\sim}{0} \quad . \tag{5.3}$$

Recalling (3.5), from (5.2), (5.3) we have

$$\eta^{(e)}(\theta,\underset{\sim}{g},0) = 0 \quad , \quad \underset{\sim}{q}^{(o)}(\theta,\underset{\sim}{0}) = \underset{\sim}{0} \quad , \quad \underset{\sim}{q}^{(e)}(\theta,\underset{\sim}{0},\dot{\theta}) = \underset{\sim}{0} \quad . \tag{5.4}$$

Now ϕ, and consequently $\eta^{(o)}$, are arbitrary functions of θ, while $\underset{\sim}{q}^{(o)}$ is any continuous function of θ, $\underset{\sim}{g}$ which is odd in $\underset{\sim}{g}$ and satisfies $(5.4)_2$. We assume that the "extra" parts of entropy and heat flux, namely $\eta^{(e)}$ and $\underset{\sim}{q}^{(e)}$, are nonlinear functions of θ, $\underset{\sim}{g}$ but are linear functions of degree one in $\dot{\theta}$ so that

$$\eta^{(e)}(\theta,\underset{\sim}{g},\dot{\theta}) = -\, n(\theta,\underset{\sim}{g})\dot{\theta} \quad , \quad \underset{\sim}{q}^{(e)}(\theta,\underset{\sim}{g},\dot{\theta}) = \underset{\sim}{\ell}(\theta,\underset{\sim}{g})\dot{\theta} \quad . \tag{5.5}$$

Assuming further that $\underset{\sim}{q}^{(o)}$, $\underset{\sim}{q}^{(e)}$ are polynomials in $\underset{\sim}{g}$ and recalling (5.4), we arrive at the forms

$$\underset{\sim}{q}^{(o)}(\theta,\underset{\sim}{g}) = -\underset{\sim}{K}(\theta,\underset{\sim}{g})\underset{\sim}{g} \quad , \quad \underset{\sim}{q}^{(e)}(\theta,\underset{\sim}{g},\dot{\theta}) = \underset{\sim}{H}(\theta,\underset{\sim}{g})\underset{\sim}{g}\dot{\theta} \quad . \tag{5.6}$$

By (3.13), (5.1) and (5.2) the functions $n(\theta,\underset{\sim}{g})$, $\underset{\sim}{K}(\theta,\underset{\sim}{g})$ in (5.5), (5.6) must satisfy

$$n(\theta,\underset{\sim}{0}) \geqq 0 \quad , \quad \underset{\sim}{K}(\theta,\underset{\sim}{g})\underset{\sim}{g} \cdot \underset{\sim}{g} \geqq 0 \quad . \tag{5.7}$$

In view of (5.6), the energy equation (3.11) now assumes the form

$$\begin{aligned}
&[-\frac{\partial}{\partial g_{\ell}}(K_{kj}\, g_j) + \frac{\partial}{\partial g_{\ell}}(H_{kj}\, g_j)\dot{\theta}]\theta_{,k\ell} + [H_{kj}\, g_j - \rho\theta\dot{\theta}\,\frac{\partial n}{\partial g_k}]\dot{\theta}_{,k} \\
&\qquad - \rho n\theta\ddot{\theta} + [-(\frac{\partial K_{kj}}{\partial\theta})g_j + (\frac{\partial H_{kj}}{\partial\theta})\dot{\theta}g_j]\theta_{,k} \\
&\qquad - \rho(n\dot{\theta} + \theta\dot{\theta}\,\frac{\partial n}{\partial\theta} + \theta\,\frac{\partial^2\psi}{\partial\theta^2})\dot{\theta} = \rho r \quad ,
\end{aligned} \tag{5.8}$$

which is a quasi-linear second order partial differential equation to be satisfied by the temperature. The classification and type of initial-boundary value problem appropriate to such an equation has been extensively discussed in the literature.*

The essential character of equation (5.8) will not be altered if we make some further simplifying assumptions. In particular, if we assume

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*See for example [15].

that n, $\underset{\sim}{K}$ and $\underset{\sim}{H}$ introduced in (5.5) and (5.6) are functions of θ only, then (5.8) becomes

$$(-K_{kj} + H_{kj}\,\dot{\theta})\theta_{,kj} + H_{kj}\,g_j\,\dot{\theta}_{,k}$$

$$-\rho n\theta\ddot{\theta} + [-(\frac{\partial K_{kj}}{\partial\theta})g_j + (\frac{\partial H_{kj}}{\partial\theta})\dot{\theta}g_j]\theta_{,k}$$

$$-\rho(n\dot{\theta} + \theta\dot{\theta}\frac{\partial n}{\partial\theta} + \theta\frac{\partial^2\psi}{\partial\theta^2})\dot{\theta} = \rho r \quad . \tag{5.9}$$

If, in addition, we assume the material is isotropic with a center of symmetry, i.e.,

$$K_{kj}(\theta) = k(\theta)\delta_{kj} \quad , \quad H_{kj}(\theta) = h(\theta)\delta_{kj} \quad , \tag{5.10}$$

then (5.9) becomes

$$(-k + h\dot{\theta})\theta_{,kk} + h\,g_k\,\dot{\theta}_{,k} - \rho n\theta\ddot{\theta}$$

$$+ (-\frac{\partial k}{\partial\theta}g_k + \frac{\partial h}{\partial\theta}\dot{\theta}\,g_k)\theta_{,k}$$

$$-\rho(n\dot{\theta} + \theta\dot{\theta}\frac{\partial n}{\partial\theta} + \theta\frac{\partial^2\psi}{\partial\theta^2})\dot{\theta} = \rho r \quad , \tag{5.11}$$

and, in view of (5.7),

$$k \geq 0 \quad . \tag{5.12}$$

We now use the one-dimensional form of (5.11) to examine more closely the propagation of thermal waves. Following the procedure that led to (4.4) we then obtain for the speed of propagation U of second order discontinuities the expression

$$U = \begin{cases} -\{h\theta_{,x} \pm [(h\theta_{,x})^2 - 4\rho n\theta(k - h\dot{\theta})]^{1/2}\}/2\rho n\theta \ , & \text{if } n \neq 0 \ , \\ (h\dot{\theta} - k)/h\theta_{,x} \ , & \text{if } n = 0 \ , \ h\theta_{,x} \neq 0 \\ \infty \ , & \text{if } n = 0 \ , \ h\theta_{,x} = 0 \ \text{(classical case)} \end{cases} \tag{5.13}$$

Equation $(5.13)_1$ reveals an interesting phenomenon. No real wave speed can result if both $\theta_{,x}$ and $\dot{\theta}$ are zero "in front of the wave," that is, if the conductor into which the wave is to propagate is in a uniform equilibrium (time-independent) thermal state. On the other hand, real wave speeds can clearly occur provided the temperature gradient in the conductor is large enough. This result appears to be in harmony with the proposals made by Chester [5] and others on a microscopic basis that thermal waves should be detectable in many solids but only above a certain critical state of thermal agitation.

6. Infinitesimal time-dependent temperature variations superposed on a finite equilibrium temperature change from reference

It was found in the previous section that a thermal wave cannot propagate into a rigid conductor with a rate-dependent thermal response unless the magnitude of the temperature gradient is sufficiently large. This fact suggests the possibility of predicting thermal waves characterized by a linear differential equation governing infinitesimal time-dependent temperature variations, provided we superpose these time-dependent variations on a finite time-independent temperature change from the constant reference temperature. Here for simplicity we confine our attention to an isotropic material and use (5.11).

Let

$$\theta(\underset{\sim}{x},t) = \tilde{\theta}(\underset{\sim}{x}) + \epsilon\theta'(\underset{\sim}{x},t) \quad , \quad r(\underset{\sim}{x},t) = \tilde{r}(\underset{\sim}{x}) + \epsilon r'(\underset{\sim}{x},t) \quad , \qquad (6.1)$$

i.e., we consider temperature variations from equilibrium that are of $O(\epsilon)$ in the limit as[+] $\epsilon \to 0$. Also let

$$\tilde{\theta}(\underset{\sim}{x}) = \theta_0 + \hat{\theta}(\underset{\sim}{x}) \quad , \qquad (6.2)$$

so that $\hat{\theta}(\underset{\sim}{x})$, which is not necessarily uniform, represents the finite time-independent variation from the reference temperature θ_0. The temperature variations $\hat{\theta}$ and θ' may be positive or negative within the limitations

[+] In this section ϵ is a small dimensionless parameter. It is implicit in (6.1) that the time and spatial derivatives of the temperature variation $\theta - \tilde{\theta}$ is also of $O(\epsilon)$.

imposed by $(6.1)_1$, (6.2) and the assumption $\theta > 0$.

Corresponding to (6.1) we introduce the following decomposition of entropy η through its parts $\eta^{(o)}$, $\eta^{(e)}$, namely

$$\begin{aligned} \eta^{(o)} &= \eta^{(o)}(\theta) = \tilde{\eta}^{(o)}(\underset{\sim}{x}) + \epsilon\eta'^{(o)}(\underset{\sim}{x},t) \quad , \\ \eta^{(e)} &= \eta^{(e)}(\theta,\underset{\sim}{g},\dot{\theta}) = \tilde{\eta}^{(e)}(\underset{\sim}{x}) + \epsilon\eta'^{(e)}(\underset{\sim}{x},t) \quad . \end{aligned} \tag{6.3}$$

From $(5.5)_1$, (6.1) and $(6.3)_2$ we find

$$\tilde{\eta}^{(e)} = 0 \quad , \quad \eta'^{(e)} = -n\dot{\theta}' \quad , \tag{6.4}$$

so that the entropy variation from its equilibrium value $\tilde{\eta}^{(o)}(\underset{\sim}{x})$ is also of $O(\epsilon)$.

Next we denote by $\tilde{\psi}(\underset{\sim}{x})$ the value of the free energy at the equilibrium state, i.e.,

$$\tilde{\psi}(\underset{\sim}{x}) = \psi[\tilde{\theta}(\underset{\sim}{x})] \quad , \tag{6.5}$$

and let $\psi'(\underset{\sim}{x},t)$ stand for the time-dependent variation of ψ from $\tilde{\psi}$ so that

$$\psi'(\underset{\sim}{x},t) = \psi(\underset{\sim}{x},t) - \tilde{\psi}(\underset{\sim}{x}) \quad . \tag{6.6}$$

Consistent with time-dependent variations in temperature and entropy that are of $O(\epsilon)$, $\psi'(\underset{\sim}{x},t)$ must be of $O(\epsilon^2)$ and it is sufficient to

assume[++]

$$\psi'(\underset{\sim}{x},t) = -\frac{1}{2}c[\epsilon\theta'(\underset{\sim}{x},t)]^2 \quad , \tag{6.7}$$

but we make no specific assumption regarding the dependence of ψ on $\tilde{\theta}$ in (6.5). Noting from $(6.1)_1$ that

$$\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\tilde{\theta}} + \frac{1}{\epsilon}\frac{\partial}{\partial\theta'} \quad , \tag{6.8}$$

we obtain from (3.9), (6.5), (6.6) and (6.7)

$$\eta^{(o)} = -\frac{\partial\tilde{\psi}}{\partial\tilde{\theta}} + c(\epsilon\theta') \quad , \tag{6.9}$$

and hence by (6.3) there results

$$\tilde{\eta}^{(o)} = -\frac{\partial\tilde{\psi}}{\partial\tilde{\theta}} \quad , \quad \eta'^{(o)} = c\theta' \quad . \tag{6.10}$$

Also, we have

$$\frac{\partial^2\psi}{\partial\theta^2} = \frac{\partial^2\tilde{\psi}}{\partial\tilde{\theta}^2} - c \quad . \tag{6.11}$$

Making use of (6.11) and (6.1) in (5.11), for an isotropic rigid conductor we obtain

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[++]We could include in (6.7) a linear term $c_1\epsilon\theta'$ but it enters the development only through its coefficient c_1 as a constant in $\tilde{\eta}^{(o)}$ and hence can be absorbed in the value of η at the reference temperature θ_o .

$$-\tilde{k}\,\tilde{\theta}_{,kk} - \tilde{k}_1\,\tilde{\theta}^2_{,k} - \rho\tilde{r} + \epsilon\{-\tilde{k}\,\theta'_{,kk} - \tilde{k}_1\,\tilde{\theta}_{,kk}\,\theta' + \tilde{h}\,\tilde{\theta}_{,kk}\,\dot{\theta}' + \tilde{h}\,\tilde{\theta}_{,k}\,\dot{\theta}'_{,k}$$

$$-\rho\,\tilde{n}\,\tilde{\theta}\,\ddot{\theta}' - \tilde{k}_1\,\tilde{\theta}_{,k}\,\theta'_{,k} - \tilde{k}_2\,\tilde{\theta}^2_{,k}\,\theta' + \tilde{h}_1\,\tilde{\theta}^2_{,k}\,\dot{\theta}'$$

$$-\rho\tilde{\theta}\Big(\frac{\partial^2\tilde{\psi}}{\partial\tilde{\theta}^2} - c\Big)\dot{\theta}' - \rho r'\} + O(\epsilon^2) = 0 \quad , \tag{6.12}$$

where $\tilde{k}$, $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{h}$, $\tilde{h}_1$ and $\tilde{n}$ are defined by the Taylor expansions

$$k(\theta) = \tilde{k} + \tilde{k}_1\,\epsilon\theta' + \frac{1}{2}\,\tilde{k}_2(\epsilon\theta')^2 + O(\epsilon^3) \quad , \quad n(\theta) = \tilde{n} + O(\epsilon)$$

$$h(\theta) = \tilde{h} + \tilde{h}_1\,\epsilon\theta' + O(\epsilon^2) \quad ; \quad \tilde{k} = k(\tilde{\theta}) \quad , \quad \tilde{k}_1 = \frac{dk}{d\theta}\bigg|_{\theta=\tilde{\theta}} \quad , \tag{6.13}$$

$$\tilde{k}_2 = \frac{d^2k}{d\theta^2}\bigg|_{\theta=\tilde{\theta}} \quad , \quad \tilde{n} = n(\tilde{\theta}) \quad , \quad \tilde{h} = h(\tilde{\theta}) \quad , \quad \tilde{h}_1 = \frac{dh}{d\theta}\bigg|_{\theta=\tilde{\theta}} \quad ,$$

assumed to be valid.

Equation (6.12) is satisfied to $O(\epsilon)$ if the finite time-independent variation $\tilde{\theta}(\underset{\sim}{x})$ satisfies

$$-\tilde{k}\,\tilde{\theta}_{,kk} - \tilde{k}_1\,\tilde{\theta}^2_{,k} = \rho\tilde{r} \quad , \tag{6.14}$$

and to $O(\epsilon^2)$ if in addition to (6.14) the infinitesimal time dependent variation $\theta'(\underset{\sim}{x},t)$ satisfies

$$-\tilde{k}\,\theta'_{,kk} + \tilde{h}\,\tilde{\theta}_{,k}\,\dot{\theta}'_{,k} - \rho\,\tilde{n}\,\tilde{\theta}\,\ddot{\theta}' - \tilde{k}_1\,\tilde{\theta}_{,k}\,\theta'_{,k} + [\tilde{h}\,\tilde{\theta}_{,kk} + \tilde{h}_1\,\tilde{\theta}^2_{,k}$$

$$+ \rho\tilde{\theta}\Big(c - \frac{\partial^2\tilde{\psi}}{\partial\tilde{\theta}^2}\Big)]\dot{\theta}' - (\tilde{k}_1\,\tilde{\theta}_{,kk} + \tilde{k}_2\,\tilde{\theta}^2_{k})\theta' = \rho r' \quad . \tag{6.15}$$

Since $\tilde{\psi}$, $\tilde{k}$, $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{n}$, $\tilde{h}$ and $\tilde{h}_1$ depend only on $\tilde{\theta}$, the coefficients in (6.15) are determined once the solution of (6.14) is known.

Equations (6.14) and (6.16) determine the infinitesimal time-dependent temperature variation superposed on a finite equilibrium change from the reference temperature when the conducting medium is characterized by the rate-dependent constitutive relations of sections 3 and 5. The corresponding results for the rate-independent response discussed in section 2, for an isotropic medium, can be obtained if we put $\tilde{n} = \tilde{h}_1 = \tilde{h}_2 = 0$. Then, (6.15) becomes

$$-\tilde{k}\,\theta'_{,kk} - \tilde{k}_1\,\tilde{\theta}_{,k}\,\theta'_{,k} + \rho\tilde{\theta}\Big(c - \frac{\partial^2\tilde{\psi}}{\partial\tilde{\theta}^2}\Big)\dot{\theta}' - \big(\tilde{k}_1\,\tilde{\theta}_{,kk} + \tilde{k}_2\,\tilde{\theta}^2_{,k}\big)\theta' = \rho r' \quad , \qquad (6.16)$$

where $\tilde{\theta}$ continues to satisfy (6.14). If, in particular, $\tilde{\theta}$ coincides with θ_o , then θ' is measured from the uniform reference temperature θ_o , $\tilde{\psi}$ will have a value ψ_o , $\tilde{r}$ must vanish according to (6.14), and (6.16) reduces to the classical form

$$-\tilde{k}\,\theta'_{,kk} + \rho\, c\, \theta_o\, \dot{\theta}' = \rho r' \quad , \qquad (6.17)$$

where $\tilde{k}$, $c\,\theta_o$ correspond to the usual thermal conductivity and specific heat.

The generalization of the heat conduction equation represented by

(6.15) is a <u>linear</u> second order partial differential equation with coefficients depending only on $\underset{\sim}{x}$ through $\tilde{\theta}$ and is in standard form.*
Its type (elliptic, parabolic, or hyperbolic) and hence the associated initial-boundary value problem that is well posed depends on the signs of its coefficients when reduced to canonical quadratic form. In particular, when $\tilde{h}$, $\tilde{n}$, $\tilde{h}_1$ vanish it reduces to the classical parabolic equation. When these coefficients do not vanish but $\tilde{\theta}_{,k}$ vanishes, i.e., when the equilibrium temperature field is uniform, then the equation is elliptic since $\tilde{k}$, $\rho\,\tilde{n}\,\tilde{\theta}$ are both non-negative. No finite speed waves are possible in either of these circumstances.

The complete classification of (6.15) for the one space dimensional case is

$$(\tilde{h}\,\tilde{\theta}_{,x})^2 - 4\rho\,\tilde{k}\,\tilde{n}\,\tilde{\theta} \begin{cases} < 0 \;\ldots\; \text{elliptic} \\ = 0 \;\ldots\; \text{parabolic} \\ > 0 \;\ldots\; \text{hyperbolic} \end{cases} . \tag{6.18}$$

The associated speeds of propagation U of second order discontinuities are

$$U = \begin{cases} \{-\tilde{h}\,\tilde{\theta}_{,x} \pm [(\tilde{h}\,\tilde{\theta}_{,x})^2 - 4\rho\,\tilde{k}\,\tilde{n}\,\tilde{\theta}]^{1/2}\}/2\rho\,\tilde{n}\,\tilde{\theta} \;, & \tilde{n} \neq 0 \\ -\tilde{k}/\tilde{h}\,\tilde{\theta}_{,x} \;, \quad \tilde{n} = 0 \;, & \tilde{h}\,\tilde{\theta}_{,x} \neq 0 \\ \infty \;, \quad \tilde{n} = 0 \;, & \tilde{h}\,\tilde{\theta}_{,x} = 0 \;. \end{cases} \tag{6.19}$$

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*For the classification of this equation and a reduction to its canonical form see [15, p. 181].

We see from (6.19) that the wave speed U must have the opposite algebraic sign from $\tilde{h}\,\tilde{\theta}_{,x}$. Hence, if $\tilde{h}$ is positive, the present theory predicts that thermal waves can propagate only in the direction from hot to cold, the same direction as the part $\underset{\sim}{q}^{(o)}$ of the heat flux. The entropy production inequality places no restriction on the sign of $\tilde{h}$. From (5.6), (5.10) we see that the "extra" part of the heat flux $\underset{\sim}{q}^{(e)}$ changes its direction with respect to the temperature gradient as the sign of $\dot{\theta}$ changes.

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This paper is concerned with conduction of heat and the related problem of propagation of thermal waves in stationary rigid solids. Special attention is given to rate-dependent response and the ensuing conditions of propagation in the conducting medium. By considering small time-dependent temperature variations superposed on a finite nonuniform equilibrium temperature field, certain conclusions are reached ~~which~~ also shed light on the so-called "second sound" phenomenon.

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